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Translated by **M.D.F.**

PMM U.S.S.R.,Vol.52,No.4,pp.518-525,1988
Printed in Great Britain
0021-8928/88 $10.00+0.00

THE METHOD OF ASYMPTOTIC INTEGRATION AND THE "METHOD OF SPRINGS" IN PROBLEMS OF ELASTIC PLATES WITH AN ELONGATED CUT*

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A class of problems in the theory of the elasticity of plates with an elongated non-through cut under arbitrary loading is analysed by the method of asymptotic integration /1-3/. An asymptotic solution in a small parameter (the ratio of the plate thickness and the length of the cut is constructed as the sum of an external solution corresponding to the two-dimensional problem of plate theory and an internal solution corresponding to the boundary layers in a zone of order h near the cut as well as the plate boundaries.

It is shown that the cut affects the elastic state of deformation in the plate (outside the boundary layers) in the second term of the external solution resulting in jumps in the kinematic and force factors on the line of the cut. Equations are obtained that express the jumps mentioned in term of the geometrical parameters of the cut and the energy characteristics of the first terms of the internal solution that is the state of plane and antiplane strain of a strip with the cut under the action of loads on the surface of the cut governed by the forces and moments of the first term of the external solution on the line of the cut. After the solution of the appropriate plane and antiplane problems for the first term of the internal solution, determination of the second term of the external solution reduces thereby to a problem in the theory of plates with the boundary conditions on the line of the cut and the edges of the plate. The second term of the asymptotic form of the boundary layer near the cut is the solution of more complex plane and antiplane problems for a strip with a cut, with a load including volume and surface forces associated with the change in the first term of the solution for the boundary layer along the cut.

Starting from the equation obtained in the case of a cut that is an extended rectilinear surface crack (normal to the plate surface) for both symmetric and antisymmetric loading approximate boundary conditions can be formulated on the line of the crack for a binomial asymptotic form of the external solution, which enables us to pose a problem in the theory of plates taking the influence of cracks into account. For symmetric loading these boundary conditions reduce to equations of the known Rice-

**Prikl.Matem.Mekhan.*,52,4,666-674,1988

Levy method of springs /4-7/ and for antisymmetric loading are new equations of similar structure, whose physical meaning is clarified in the paper. the stress intensity factors on its contour can be determined approximately from the force and moments on the line of cracks found by using these equations.

The problem of a surface crack in a unbounded plate under antisymmetric loading is considered as an example. This reduces to a system of two singular integral equations, which can be solved numerically. Results are presented of a computation for a semi-elliptical shape.

1. Consider a plate given in Cartesian coordinates x_i by the relations $(x_1, x_2) \in D$, $|x_3| \leqslant h/2$ (D is a certain plane domain). Let $R(l)$ be a plane curve of length $2L$ within the domain D (Fig.1), and let the parameter $l \in [-L, L]$ be the spacing along the curve from its centre $L \gg h$. We introduce an orthonormal triplet of vectors $\mathbf{t}(l) = d\mathbf{R}(l)/dl$, $\mathbf{v}(l) = \mathbf{e}_3 \times \mathbf{t}(l)$, $\mathbf{e}_3$, a curvilinear orthogonal coordinate system l, m, x_3, i.e., $\mathbf{x}(l, m, x_3) = \mathbf{R}(l) + m\mathbf{v}(l) + x_3\mathbf{e}_3$ and also the dimensionless coordinates

$$X_i = \frac{x_i}{L}\ (i = 1, 2),\quad X_3 = Z = \frac{2x_3}{h},\quad x = \frac{l}{L},\quad y = \frac{m}{h},\quad Y = \frac{2m}{h}$$

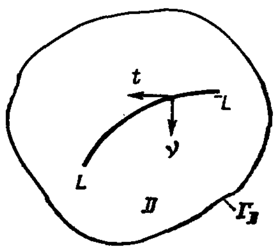

Fig.1

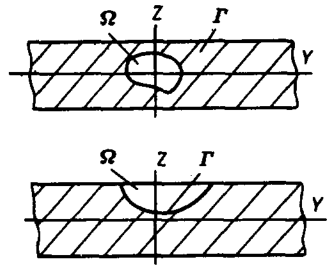

Fig.2

We assume that the plate contains an elongated cut Ω (internal or surface) that occupies a domain determined by the relationships $f(x, Y, Z) \leqslant 0$, $|x| \leqslant 1$, $|Z| \leqslant 1$ in x, Y, Z coordinates (f is a fairly smooth function) so that in each section $x = x_0$ the domain of the cut

$$\Omega(x_0) = \{(Y, Z) \mid f(x_0, Y, Z) \leqslant 0, |Z| \leqslant 1\}$$

is internal or adjoins $Z = 1$ (Fig.2). Let the plate surfaces $Z = \pm 1$ be loaded by the pressure $p^{\pm}(x_1, x_2)$, respectively, while the surface Γ of the cut in the case of an internal cut is force-free, and is loaded by the pressure $p^{+}(x_1, x_2)$ in the case of an edge cut. Certain boundary conditions are given on Γ_D (their exact form is not important here).

We consider the asymptotic behaviour of the solution of the problem mentioned in the small parameter $\varepsilon = h/2L$ (actually, we limit ourselves to the first two terms of the asymptotic form in this paper).

We introduce the dimensionless stress and displacement

$$\sigma_{ij}^{\circ} = \sigma_{ij}/\sigma_0,\quad u_i^{\circ} = 2\mu u_i/(\sigma_0 L)$$

where σ_0 is the characteristic quantity $p^{+}(x_1, x_2)$ and μ is the shear modulus.

We will construct the asymptotic form as

$$u_i^{\circ} = u_i^{p} + u_i^{b} + u_i^{d},\quad \sigma_{ij}^{\circ} = \sigma_{ij}^{p} + \sigma_{ij}^{b} + \sigma_{ij}^{d} \tag{1.1}$$

where u_i^p, σ_{ij}^p are the dimensionless displacement and stress far from the edge Γ_D and the cut, $u_i^b, \sigma_{ij}^b, u_i^d, \sigma_{ij}^d$ are the displacement and stress of edge effects near Γ_D and the cut important at distances of the order of h from them and damping exponentially with distance from Γ_D and Ω.

In the general case, the edge effect near the ends of the cut should also be taken into account; however we shall consider these ends as not too "blunt" in this paper so that the influence of the mentioned edge effect in the terms of the asymptotic form under consideration need not be taken into account far from the cut and near its centre part.

The method of constructing the quantities u_i^b, σ_{ij}^b and the refined boundary conditions in u_i^p, σ_{ij}^p at the plate edges obtained by taking them into account is known /1-3/. In this paper we examine the method of constructing the quantities u_i^d, σ_{ij}^d and the corresponding conditions for u_i^p, σ_{ij}^p on the line of the cut. When solving problems of bounded plates with cuts here it is necessary to use the equations obtained below in conjunction with the boundary conditions /1-3/; it is not required to use these in the problem of an unbounded plate with a cut (see Sect.3).

The asymptotic form u_i^p, σ_{ij}^p of an elastic field in a plate far from inhomogeneities has the form /7/

$$u_i{}^p = \varepsilon^{\beta_i}\Sigma V_i^{(n)}(X_1, X_2, X_3)\,\varepsilon^n, \quad \sigma_{ij}{}^p = \varepsilon^{k_{ij}}\Sigma S_{ij}^{(n)}(X_1, X_2, X_3)\,\varepsilon^n \tag{1.2}$$
$$\beta_1 = \beta_2 = -2,\ \beta_3 = -3,\ k_{12} = k_{11} = k_{22} = -2,\ k_{13} = k_{23} = -1,$$
$$k_{33} = 0$$

(here and henceforth the summation is over n between $n = 0$ and $n = \infty$) and should satisfy the equilibrium equations $S_{ij,j}^{(n)} = 0$, the Hooke's law relationships, and the conditions on the surfaces $X_3 = \pm 1$. Using these conditions, it can be shown in particular that the first two terms of the asymptotic form satisfy relationships of the theory of plate bending and tension (ν is Poisson's ratio)

$$V_3^{(n)} = w^{(n)}(X_1, X_2), \quad V_i^{(n)} = v_i^{(n)}(X_1, X_2) - Zw_{,i}^{(n)} \tag{1.3}$$
$$S_{ii}^{(n)} = n_{ii}^{(n)} + Zm_{ii}^{(n)}, \quad S_{ij}^{(n)} = n_{ij}^{(n)} + Zm_{ij}^{(n)}, \quad S_{i3}^{(n)} = {}^1/_2(1 - Z^2)\,q_i^{(n)}$$
$$n_{ii}^{(n)} = (1-\nu)^{-1}(v_{i,i}^{(n)} + \nu v_{j,j}^{(n)}), \quad m_{ii}^{(n)} = -(1-\nu)^{-1}(w_{,ii}^{(n)} + \nu w_{,jj}^{(n)})$$
$$n_{ij}^{(n)} = {}^1/_2(v_{i,j}^{(n)} + v_{j,i}^{(n)}), \quad m_{ij}^{(n)} = -w_{,ij}^{(n)}, \quad q_i^{(n)} = -(1-\nu)^{-1}\nabla^2 w_{,i}^{(n)}$$
$$S_{33}{}^{(0)} = {}^1/_2\,(S_+ + S_-) + {}^3/_4\,(S_+ - S_-)(Z - {}^1/_3 Z^3),\ S_{33}{}^{(1)} = 0$$
$$(1+\nu)(1-\nu)^{-1}(v_{i,i}^{(n)} + v_{j,j}^{(n)})_{,i} + \nabla^2 v_i{}^{(n)} = 0$$
$$\nabla^4 w^{(0)} = {}^3/_2\,(1-\nu)(S_+ - S_-),\ \nabla^4 w^{(1)} = 0$$
$$(n = 1, 2;\ (i, j) = (1,2) \text{ or } (2,1);\ S_\pm = p^\pm/\sigma_0).$$

Since the forces are given on the surface of the cut, the stresses for the boundary layer near the cut should be of the same order as the stresses for the external solution (1.2), i.e., $\varepsilon^{-2}\sigma_0$. Since the characteristic linear dimension of the boundary layer equals h, the displacements for the boundary layer should be of the order of $\varepsilon^{-2}\sigma_0 h\mu^{-1} = 2L\mu^{-1}\sigma_0\varepsilon^{-1}$. Consequently we seek the asymptotic form $\sigma_{ij}{}^d$, $u_i{}^d$ as

$$u_i{}^d = \varepsilon^{-1}\Sigma u_i^{(n)}(x, Y, Z)\,\varepsilon^n, \quad \sigma_{ij}{}^d = \varepsilon^{-2}\Sigma\sigma_{ij}^{(n)}(x, Y, Z)\,\varepsilon^n \tag{1.4}$$

Terms of the expansion (1.4) should satisfy (asymptotically) the equilibrium equations and the Hooke's law relationships /2/, as well as homogeneous equations on the plate surfaces

$$\sigma_{xZ}^{(n)}\ (x, Y, \pm 1) = \sigma_{YZ}^{(n)}\ (x, Y, \pm 1) = \sigma_{ZZ}^{(n)}\ (x, Y, \pm 1) = 0$$

and vanish at infinity ($u_i{}^d \to 0$, $\sigma_{ij}{}^d \to 0$ as $|Y| \to \infty$). Moreover, the field $\sigma_{ij}{}^d + \sigma_{ij}{}^p$ should asymptotically yield given forces on the surface Γ of the cut. Finally, since the components of the expansion (1.2) can generally experience a discontinuity when passing through the surface $y = 0$ (corresponding to the cut), it must be required that $u_i{}^d + u_i{}^p$, $\sigma_{ij}{}^d + \sigma_{ij}{}^p$ asymptotically satisfy the force and displacement continuity conditions when passing through the surface $Y = 0$ outside the cut.

Taking into account the relationships given above, the problem for each term of the asymptotic form (1.4) can (as in /2/ be reduced for each fixed $x = x_0$ to the plane and antiplane problems of an infinite strip with a cut $\Omega\,(x_0)$ loaded by certain surface and volume forces expressed in terms of the previous terms of the asymptotic form (1.4) and terms of the asymptotic form (1.2). The conditions at the jumps of the terms of the asymptotic form (1.2) are here selected for $y = 0$ so as to ensure the existence of solutions of the corresponding plane and antiplane problems as well as the continuity of the displacements and forces for $Y = 0$.

In accordance with the above we consider the relationships for the first terms of the asymptotic forms (1.2) and (1.4). Since the order of the first term of the asymptotic form $u_i{}^p$ is greater than the order of the asymptotic form $u_i{}^d$, it follows from the condition of continuity for the displacement that the function $V_i{}^{(0)}\,(x, y, Z)$ is continuous for $y = 0$, i.e., the jumps in the displacements and the angles of rotation of the middle surface are zero for $y = 0$:

$$\Delta v_x{}^{(0)} = \Delta v_y{}^{(0)} = \Delta w^{(0)} = \Delta w_{,y}{}^{(0)} = 0 \tag{1.5}$$

The conditions at the jump $S_{ij}{}^{(0)}$ are obtained when considering the plane and antiplane problems for $\sigma_{ij}{}^{(n)}$ for $n = 0$ and 1. For the existence of a solution that damps out at infinity for these problems, the external forces applied to a half-strip with a cut should be balanced, i.e., for any x the following quantities should equal zero:

1) The principal vector of the body and surface forces in the antiplane problem $Q_x{}^{(n)}$;

2) The principal vector of the body and surface forces in the plane problem $Q_Y{}^{(n)}\,\mathbf{v}\,(x) + Q_Z{}^{(n)}\mathbf{e}_3$;

3) The component in $\mathbf{t}\,(x)$ of the principal moment of the body and surface forces in the plane problem $M_x{}^{(n)}$.

It can be verified that the relationships $Q_Z^{(0)} = 0$ are satisfied automatically, while the relationships $Q_Y^{(0)} = Q_x^{(0)} = M_x^{(0)} = 0$ determine the zero jumps of the forces $n_{xy}^{(0)}$, $n_{yy}^{(0)}$ and the moment $m_{yy}^{(0)}$. Since the stress from the transverse forces is an order of magnitude smaller than the shear stress the jump in the transverse force $q_y^{(0)}$ is found from the condition $Q_Z^{(1)} = 0$. Hence

$$\Delta n_{xy}^{(0)} = \Delta n_{yy}^{(0)} = \Delta m_{yy}^{(0)} = \Delta q_y^{(0)} + \Delta m_{xy,\,x}^{(0)} = 0 \tag{1.6}$$

Relations (1.5) and (1.6) shows that $V_i^{(0)}$ and $S_{ij}^{(0)}$ are continuous for $y = 0$ and are determined just by relations (1.3) and the boundary conditions on Γ_D, i.e., the elastic field in the plate to a first approximation "does not notice" the cut.

Taking (1.6) into account, we reduce the plane and antiplane problems for the stress $\sigma_{ij}^{(0)}$ for each x to the determination of the stresses that vanish at infinity $\sigma_{ij}^* = \sigma_{ij}^{(0)}$ in a strip with a cut loaded only on the surface $\Gamma(x)$ of the cut and such that the forces on the surface $\Gamma(x)$ compensate the forces from $S_{ij}^{(0)}$

$$\begin{gathered}\sigma_{xY}^* n_Y + \sigma_{xZ}^* n_Z = -S_{xy}^{(0)}(x,0,Z)\, n_Y,\quad \sigma_{YY}^* n_Y + \sigma_{YZ}^* n_Z = -S_{yy}^{(0)}(x,0,Z)\, n_Y \\ \sigma_{ZY}^* n_Y + \sigma_{ZZ}^* n_Z = 0 \quad (Y,Z) \in \Gamma(x) \\ S_{\alpha y}^{(0)}(x,0,Z) = n_{\alpha y}^{(0)}(x,0) + Z m_{\alpha y}^{(0)}(x,0) \quad (\alpha = x, y)\end{gathered} \tag{1.7}$$

Since $\sigma_{ij}^* \to 0$ as $Y \to \pm\infty$, we have for the displacement u_i^* in the corresponding plane and antiplane problem

$$\begin{gathered}u_x^*(x,Y,Z) = u_x^{\pm\infty} + o(1),\quad u_Y^*(x,Y,Z) = u_Y^{\pm\infty} + Z\theta^{\pm\infty} + o(1) \\ u_Z^*(x,Y,Z) = u_Z^{\pm\infty} - Y\theta^{\pm\infty} + o(1),\ Y \to \pm\infty\end{gathered}$$

while the displacements $u_i^{(0)}$ are expressed in terms of u_i^* as follows:

$$\begin{gathered}u_x^{(0)} = u_x^* - u_x^{\lambda},\quad u_Y^{(0)} = u_Y^* - u_Y^{\lambda} - Z\theta^{\lambda}, \\ u_Z^{(0)} = u_Z^* - u_Z^{\lambda} + Y\theta^{\lambda} \\ \lambda = +\infty,\ Y > 0;\ \lambda = -\infty,\ Y < 0\end{gathered} \tag{1.8}$$

and are defined uniquely (while u_i^* are determined to the accuracy of a rigid displacement and a rotation), and satisfy the condition $u_i^{(0)} \to 0$ as $Y \to \pm\infty$.

The condition of continuity $u_i^d + u_i^p$ for $Y = 0$ outside Ω for terms of the order of ε^{-2} and ε^{-1} is sought taking (1.8) into account in the form

$$\begin{gathered}\Delta w^{(1)} = 0,\quad \Delta v_x^{(1)} = \{u_x\}_\infty,\quad \Delta v_y^{(1)} = \{u_Y\}_\infty,\quad \Delta w_{,y}^{(1)} = -\{\theta\}_\infty, \\ \Delta w^{(2)} = \{u_Z\}_\infty \\ \{u_\alpha\}_\infty = u_\alpha^{+\infty} - u_\alpha^{-\infty},\quad \{\theta\}_\infty = \theta^{+\infty} - \theta^{-\infty} \quad (\alpha = x, Y, Z)\end{gathered} \tag{1.9}$$

i.e., the jumps in the displacements and the angles of rotation are expressed in terms of the relative displacements and rotations of the ends of the strip.

The quantities $\{u_x\}_\infty$, $\{u_Y\}_\infty$, $\{\theta\}_\infty$ can be expressed in terms of the power and geometrical characteristics of a strip with a cut. To do this it is sufficient to write the relationship of the Betti reciprocity theorem for an elastic field u_i^*, σ_{ij}^* and, respectively, for the fields of a homogeneous antiplane shear, a homogeneous plane strain, and pure bending of the strip. Therefore the following relationships are easily obtained:

$$\begin{gathered}\Delta v_x^{(1)} = \{u_x\}_\infty = U_x^n + W_x^n \\ \Delta v_y^{(1)} = \{u_Y\}_\infty = U_y^n + {}^1/_2(1-\nu)\, W_y^n \\ \Delta w^{(1)} = 0,\quad -{}^1/_3 \Delta w_{,y}^{(1)} = {}^1/_3\{\theta\}_\infty = U_y^m + {}^1/_2(1-\nu)\, W_y^m\end{gathered} \tag{1.10}$$

Here

$$\begin{gathered}\begin{Vmatrix} U_\alpha^n \\ U_\alpha^m \end{Vmatrix} = \begin{Vmatrix} A_\alpha^{nn} & A_\alpha^{nm} \\ A_\alpha^{nm} & A_\alpha^{mm} \end{Vmatrix} \begin{Vmatrix} n_{\alpha y}^{(0)} \\ m_{\alpha y}^{(0)} \end{Vmatrix} \\ \begin{Vmatrix} W_\alpha^n \\ W_\alpha^m \end{Vmatrix} = \begin{Vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{Vmatrix} \begin{Vmatrix} n_{\alpha y}^{(0)} \\ m_{\alpha y}^{(0)} \end{Vmatrix} \\ W_k(x) = \iint\limits_{\Omega(x)} Z^k\, dY\, dZ,\quad A_\alpha^{nn}(x) = \frac{1}{2} \int\limits_{\Gamma(x)} u_\alpha^n n_Y\, dS\end{gathered}$$

$$A_\alpha^{nm}(x) = \frac{1}{2}\int_{\Gamma(x)} u_\alpha{}^m n_Y\, dS = \frac{1}{2}\int_{\Gamma(x)} Z u_\alpha{}^n n_Y\, dS,$$

$$A_\alpha^{mm}(x) = \frac{1}{2}\int_{\Gamma(x)} Z u_\alpha{}^m n_Y\, dS \quad (\alpha = x, y)$$

Values of the forces and moments are taken for $y = 0$; the quantities $u_\alpha{}^n$, $u_\alpha{}^m$ $(\alpha = x, y)$ are the displacement $u_\alpha{}^*$ for a load (1.7) on $\Gamma(x)$ corresponding to the cases of unit force $(n_{\alpha y}{}^{(0)}(x, 0, Z) = 1,\ m_{\alpha y}{}^{(0)}(x, 0, Z) = 0)$ and unit moment $(n_{\alpha y}{}^{(0)}(x, 0, Z) = 0,\ m_{\alpha y}{}^{(0)}(x, 0, Z) = 1)$. Therefore, the coefficients W and A depend only on the shape and location of the cut $\Omega(x)$ (and the coefficients $A_y{}^{nn}$, $A_y{}^{nm}$, $A_y{}^{mm}$ on Poisson's ratio ν also), and have the following meaning: $W_k(x)$ are the geometric moments of the domain $\Omega(x)$ and A is the dimensionless energy (mutual energy) of the solutions of the plane (antiplane) problems about a strip with a cutout loaded on $\Gamma(x)$ by forces corresponding to a unit force and moment.

To close the problem for the second term of the asymptotic form (1.2), four more relations must still be obtained on the jumps of the forces and moments for $y = 0$. They are obtained from the conditions $Q_x{}^{(1)} = Q_Y{}^{(1)} = Q_Z{}^{(2)} = M_x{}^{(1)} = 0$ and can, after rather lengthy reduction, be reduced to the following form:

$$\begin{aligned}
&\Delta n_{xy}^{(1)}\mathbf{t} + \Delta n_{yy}^{(1)}\mathbf{v} = [W_*{}^n\mathbf{t}]_{,x} \qquad (1.11)\\
&{}^1/_3\Delta m_{yy}^{(1)} = [U_x{}^m + W_x{}^m]_{,x} - k(x)W_*{}^m\\
&{}^1/_3[\Delta q_y^{(1)} + \Delta m_{xy,x}^{(1)}] = [k(x)(U_x{}^m + W_x{}^m) + W_{*,x}^m]_{,x}\\
&\begin{Vmatrix} W_*{}^n \\ W_*{}^m \end{Vmatrix} = \begin{Vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{Vmatrix}\begin{Vmatrix} n_{xx}^{(0)} - \nu n_{yy}^{(0)} \\ m_{xx}^{(0)} - \nu m_{yy}^{(0)} \end{Vmatrix}
\end{aligned}$$

($k(x)$ is the dimensionless curvature of the curve $R(x)$: $d\mathbf{t}/dx = -k(x)\mathbf{v}$).

After solving the problem of plate theory for $V_i{}^{(1)}$ and $S_{ij}{}^{(1)}$, defined by relations (1.3), (1.10), (1.11) and the boundary conditions on Γ_D, the problem for the second term of the asymptotic form (1.4) can be formulated completely. The relations obtained here are not presented because of their complexity.

2. In the special limiting case of the problem in Sect.1, when a rectilinear surface crack normal to the surface $Z = 1$ that occupies the domain $|x| \leqslant 1,\ Y = 0,\ 1 - 2\zeta(x) \leqslant Z \leqslant 1$ $(0 \leqslant \zeta \leqslant 1)$, is taken as the cut, the formulas in Sect.1 simplify considerably. We introduce dimensionless forces, moments and displacements of the middle surface while restricting ourselves to two terms of the asymptotic form

$$v_x = v_x^{(0)} + \varepsilon v_x^{(1)},\quad v_y = v_y^{(0)} + \varepsilon v_y^{(1)},\quad w = w^{(0)} + \varepsilon w^{(1)},$$
$$n_{ij} = n_{ij}^{(0)} + \varepsilon n_{ij}^{(1)} \quad \text{etc.}$$

and we write relations (1.5), (1.6), (1.10) and (1.11) in terms of them while neglecting terms of the order of ε^2. We here represent the state of stress and strain of a plate as the sum of symmetric and antisymmetric components relative to the $y = 0$ plane and we consider each of them separately.

Symmetric loading. In this case the relationships of Sect.1 yield on the line $y = X_2 = 0$

$$\begin{aligned}
&\Delta v_y = \varepsilon(A_y^{nn}n_{yy} + A_y^{nm}m_{yy}) \qquad (2.1)\\
&-{}^1/_3\Delta w_{,y} = \varepsilon(A_y^{nm}n_{yy} + A_y^{mm}m_{yy})\\
&\Delta v_x = \Delta w = \Delta m_{yy} = \Delta n_{yy} = n_{xy} = q_y + m_{xy,x} = 0
\end{aligned}$$

i.e., the equations of the Rice-Levy "spring model". In this case the coefficients A_y^{nn}, A_y^{nm}, A_y^{mm} as functions of ζ and ν have been evaluated and tabulated /6/.

Relationships for the stress intensity factors (to ε^2 accuracy)

$$\begin{aligned}
&K_{\mathrm{I}}/K^\circ = K_{\mathrm{I}}{}^n(\zeta(x))\, n_{yy}(x, 0) + K_{\mathrm{I}}{}^m(\zeta(x))\, m_{yy}(x, 0) \qquad (2.2)\\
&K_{\mathrm{II}} = K_{\mathrm{III}} = 0 \quad (K^\circ = \varepsilon^{-2}\sigma_0\sqrt{\pi h\zeta(x)})
\end{aligned}$$

can be obtained from the relations for the first two terms of the asymptotic form (1.4), where the dimensionless intensity factors $K_{\mathrm{I}}{}^n$, $K_{\mathrm{I}}{}^m$ are known /8/ in the plane problem of a strip with an edge crack loaded by a unit constant and linear load.

Antisymmetric loading. In this case the relationships of Sect.1 on the line $y = X_2 = 0$ yield

$$\begin{aligned}
&\Delta v_x = \varepsilon(A_x^{nn}n_{xy} + A_x^{nm}m_{xy}) \qquad (2.3)\\
&{}^1/_3\Delta m_{yy} = \varepsilon(A_x^{nm}n_{xy} + A_x^{mm}m_{xy})_{,x}
\end{aligned}$$

$$\Delta v_y = \Delta w_{,y} = \Delta n_{xy} = \Delta q_y + \Delta m_{xy,x} = w = n_{yy} = 0$$

Eqs.(2.3) show that under antisymmetric loading a surface crack can be modelled by a through crack with linear elastic connections between the edges and an additionally applied bending moment.

The difference between (2.1) and (2.3) can be clarified as follows. Eqs.(2.1) are obtained for the symmetric case if it is assumed /4/ that the elastic field near the crack is a plane strain (solution of the problem of the tension and bending of a plane strip with a notch at each section $x = x_0$), and the additional displacement and rotation of the ends of the strip relative to each other that occur because of the presence of the notch are taken into account in determining the far field. Analogously, (2.3) can be obtained if it is considered that the elastic field near the crack is antiplane strain (the solution of the problem of the antiplane shear of a strip with a notch) and the additional relative displacement of the strip ends (the first equation in (2.3)) is taken into account in determining the far field. However, in this case it is also necessary to take into account that, unlike the symmetric case, the domain adjoining the crack is not balanced completely for stresses selected by the mode mentioned.

Indeed, if a thin layer $|Z| \leqslant 1$, $x_0 \leqslant x \leqslant x_0 + \Delta x$, containing part of the crack is examined, it can be shown that the additional elastic field due to the notch yields forces whose total moments about the x axis differ from zero on the surfaces $x = x_0$, $x = x_0 + \Delta x$. Since the additional elastic field depends on x, we obtain that a non-zero moment acts on the layer under consideration, which should be balanced at infinity, i.e., from the far field side. This means that the latter should have an appropriate jump in the bending moment on the crack, which is the second equation of (2.3).

In this case we have for the stress intensity factors

$$\begin{aligned} &K_{\rm I} = 0,\ K_{\rm II}/K^\circ = o(\varepsilon) \\ &K_{\rm III}/K^\circ = K_{\rm III}^n(\zeta(x))n_{xy}(x,0) + K_{\rm III}^m(\zeta(x))m_{xy}(x,0) \end{aligned} \tag{2.4}$$

where $K_{\rm III}^n, K_{\rm III}^m$ are factors analogous to $K_{\rm I}^n, K_{\rm I}^m$, the factors for the antiplane problem. From (2.4) we have for the specific energy increment during crack growth

$$\delta W/\delta S = {}^1/_2\mu\,(K^\circ)^2\,[(K_{\rm III}^n n_{xy} + K_{\rm III}^m m_{xy})^2 + o(\varepsilon)] \tag{2.5}$$

The quantities

$$A_x^{nn}, A_x^{nm}, A_x^{mm}, K_{\rm III}^n, K_{\rm III}^m \tag{2.6}$$

can be found by solving the appropriate antiplane problem by methods of the theory of functions of a complex variable. We consequently obtain

$$\begin{aligned} &K_{\rm III}^n = \left[\frac{2}{\pi\zeta}\,\mathrm{tg}\,\frac{\pi}{2}\zeta\right]^{1/2}, \quad K_{\rm III}^m = F\left(\sin\frac{\pi}{2}\zeta\right)K_{\rm III}^n \\ &A_x^{nn}(\zeta) = G_0(\zeta) = -\frac{16}{\pi}\ln\cos\frac{\pi}{2}\zeta, \quad A_x^{nm}(\zeta) = G_1(\zeta), \\ &A_x^{mm}(\zeta) = G_2(\zeta) \\ &G_k = 8\int_0^\zeta \mathrm{tg}\,\frac{\pi}{2}t F^k\left(\sin\frac{\pi}{2}t\right)dt \quad (k = 0, 1, 2), \\ &F(t) = 1 - \frac{8}{\pi^2}\Sigma\frac{t^{2n+1}}{(2n+1)^2} = 1 - \frac{4}{\pi^2}[\mathrm{Li}_2(t) - \mathrm{Li}_2(-t)], \\ &\mathrm{Li}_2(t) = -\int_0^t \frac{\ln(1-\alpha)}{\alpha}d\alpha \end{aligned} \tag{2.7}$$

Fig.3 shows graphs of the quantities (2.6) as a function of ζ (the dashed curves *1-3* and the solid curves *1* and *2*, respectively).

3. As an example of the application of Eqs.(2.3) presented in Sect.2, we consider the problem of an edge crack in an infinite plate under antisymmetric loading. In this case the elastic field is the sum of a uniform field that would occur in a plate without the crack (denoted below by the superscript ∞) and the perturbation from a crack that damps out at infinity (denoted by the superscript c). The first field is continuous for $y = 0$, while for the second system (2.3) is written in the form

$$\begin{aligned} &\Delta v_x^c = \varepsilon\,(A_x^{nn}n_{xy} + A_x^{nm}m_{xy}) \\ &{}^1/_3\Delta m_{yy}^c = \varepsilon\,(A_x^{nm}n_{xy} + A_x^{mm}m_{xy})_{,x} \\ &\Delta v_y^c = \Delta w_{,y}^c = \Delta n_{xy}^c = \Delta q_y^c + \Delta m_{xy,x}^c = w^c = n_{yy}^c = 0 \end{aligned} \tag{3.1}$$

Solving the problem of the theory of plate tension and beding given by (1.3) and (3.1) for the perturbation field from the crack, it can be shown that

$$n_{xy}^c(x,0)=I, \quad m_{xy}^c(x,0)=-\frac{1-\nu}{4\pi}\int_{-1}^{1}\frac{\Delta m_{yy}^c(x')}{x'-x}dx'=-J \tag{3.2}$$
$$I=\frac{1+\nu}{4\pi}\int_{-1}^{1}\frac{\Delta v_x{}^c(x')_{,x'}}{x'-x}dx', \quad J=\frac{3(1-\nu)}{4\pi}\int_{-1}^{1}\frac{\delta(x')_{,x'}}{x'-x}dx'$$

where the integrals are understood in the principal value sense, the notation

$$\delta(x)=\varepsilon(A_x^{nm}n_{xy}+A_x^{mm}m_{xy}) \tag{3.3}$$

is introduced, and the second relationship of (3.1) is used. Taking account of (3.1) and (3.2) the problem reduces to a system of singular integral equations whose form is analogous to the corresponding equations of the method of springs /6/

$$\Delta v_x{}^c(x)-\varepsilon IA_x^{nn}(\zeta(x))+\varepsilon JA^{nm}(\zeta(x))= \varepsilon[A_x^{nn}(\zeta(x))n_{xy}^\infty(x,0)+A_x^{nm}(\zeta(x))m_{xy}^\infty(x,0)] \tag{3.4}$$
$$\delta(x)-\varepsilon IA_x^{nm}(\zeta(x))+\varepsilon JA_x^{mm}(\zeta(x))= \varepsilon[A_x^{nm}(\zeta(x))n_{xy}^\infty(x,0)+A_x^{mm}(\zeta(x))m_{xy}^\infty(x,0)]$$

Therefore, if the stresses are known in a plate without a crack, the problem reduces to solving two singular Eqs.(3.4) which can be solved numerically (by the method of mechanical quadratures /8/, say), after which the quantities K_{III} and $\delta W/\delta S$ governing the possibility and growth rate of the crack in the case under consideration can be found from (2.4) and (2.5) by finding the quantities $n_{xy}(x,0)$ and $m_{xy}(x,0)$ from (3.3) and the first equation of (3.1).

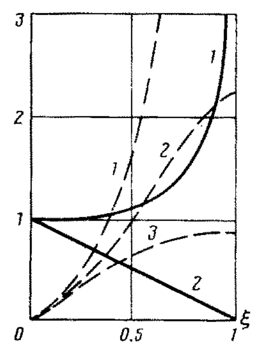

Fig.3

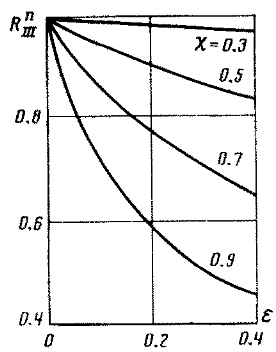

Fig.4

As a specific example we considered a semi-elliptical crack $\zeta(x)=\chi\sqrt{1-x^2}$ under a uniform shear load $(n_{xy}{}^\infty(x,0)=n_{xy}{}^\infty=\text{const},\ m_{xy}{}^\infty(x,0)=0)$. Eqs.(3.4) were solved approximately by the method of mechanical quadratures on a computer for different values of ε and χ characterizing the elongation and relative depth of the crack. According to (2.4) and (2.7), we here have $K_I=K_{II}=0$ (from the symmetry of the problem) at the apparently most dangerous point of the contour corresponding to $x=0$ (from the viewpoint of crack growth), and

$$K_{III}=K^\circ K_{III}^n(\chi)R_{III}^n(\varepsilon,\chi,\nu)n_{xy}{}^\infty(0,0) \tag{3.5}$$
$$(R_{III}^n(\varepsilon,\chi,\nu)=[n_{xy}(0,0)+m_{xy}(0,0)F(\sin{}^1/_2\pi\chi)]/n_{xy}{}^\infty(0,0))$$

where R_{III}^n is a dimensionless coefficient reflecting the influence of the spatial geometry of the problem. Graphs of its dependence on ε for $\nu=0.3$ are shown in Fig.4.

The results of calculations show that as the crack elongation decreases (i.e., as ε grows), the intensity factor K_{III} decreases, where for sufficiently deep cracks it is essential that the strength of structures with cracks of the kind under consideration should be taken into account in the calculations. Only cracks of quite small depth ($\chi\approx 0.1$ and less) are an exception for which a calculation shows that because of the action of the bending moment m_{xy} due to the presence of the crack the factor K_{III} increases insignificantly (fractions of a percent) as ε increases.

The results obtained for not too "shallow" cracks are quite similar to the results for separation cracks /5-7/, by as χ increases the factor K_{III} decreases rather more slowly than K_I as compared with the case of a separation crack.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.52,No.4,pp.525-533,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

THERMOELASTIC STRESSES IN A HALF-SPACE HEATED BY A CONCENTRATED ENERGY FLUX*

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An exact solution is obtained for the problem and also a simple approximate solution convenient for computations for small times (its error is estimated) that is valid for any absorption coefficients. In the special case of a zero absorption coefficient, the solution is simplified and can be written in elementary functions (an example is presented). In this case new qualitative features of the stress field are found that are not inherent in other methods of heating the half-space. For fairly large absorption coefficients (a criterion is given), a still more simple and convenient closed solution for computations is successfully obtained which can also be expressed in terms of elementary functions (an example is presented). In the case of both large and small absorption coefficients the stress field is analysed and its isolines are constructed.

In a number of cases, temperature stresses that can be the cause of brittle fracture /1-5/ can occur in a solid subjected to a constant energy flux (a laser beam, an electron beam, etc.). The temperature stresses in the body under exposure are studied below on the basis of the extensively utilized model of an elastic half-space (/2-7/, say). It is assumed that internal distributed heat sources whose density decreases exponentially with depth (Bouger's law /5, 8/) act in the half-space. Convective heat transfer from a zero-temperature medium occurs on the half-space boundary. This model is quite adequate and allows a determination of the thermoelastic stresses at both great depths and at depths of the order of the characteristic absorption scale or less.

The plane thermoelasticity problem for a half-space with heat sources was solved in /9/. However, real high-energy beams ordinarily possess axial symmetry. The temperature and thermoelastic stresses in the half-space in /3/ were found by numerical integration of two improper integrals, in the form of which the exact solution is represented, for the case of a uniform energy distribution over the transverse section of a cylindrical beam. An attempt to construct

**Prikl.Matem.Mekhan.*,52,4,675-684,1988